INVESTMENT AND FINANCE: OPTIMIZATION OVER TIME IN A STOCHASTIC SETUP

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INTRODUCTION

This paper is a study on the interaction between investment and financial decisions at the level of the firm. The problem is studied in a dynamic setting with irreversible investment and uncertainty about future prices of output of real wages and of interest rates. The physical capital of the firm is an entity which incorporates the technological organizational and entrepreneurial characteristics which are unique to the firm and which determine it profitability. Other features of the study are (a) the existence of a Stock Market in which shares of firms are traded. Shares are, in our context, titles of ownership of a proportion of the capital stock of firms; (b) the recognition of the fact that firms can influence the price of their shares. This follows from the unique characteristics of the capital assets of each firm, and our definition of shares; (c) the distinction between ownership and control of firms; (d) the realization that increasing the productive capacity of a firm entails additional costs in the form of better organization, marketing etc. This is in the spirit of "installation costs"; a concept which has been already integrated into the theory of investment.

We will comment briefly on the features of our model and their relation to economic literature.

Until recently economic theory neglected the interrelation between "real" and "financial" decisions of the firm. Models set in a stochastic environment dealth with either production-investment

problems, or with the equilibrium prices of financial assets and the properties of optimal portfolios. Lately "real" decisions have been incorporated into models of financial equilibrium [3], [5], [9]. Furthermore the role of the stock market has been explicitly considered [5], [9], [6], The recent literature has contributed considerably in the integration of real and financial theories. Its richness , though, is somehow diminished for two reasons; either (i) the basic decision making criteria of most models have been convincingly challenged or (ii) the models are static and do not readily lend themselves to dynamic extentions (as it is inherent in the nature of investment decisions)[6]. We will discuss the first reason in greater detail.

When the outcome of investment is certain and capital markets are perfect, the value of a share is the same to all investors and equal to its current market price. Under these conditions the investment policy of the management is guided, and ultimately evaluated by its effect, on the market price of shares. Furthermore, real and financial decisions are independent of each other.

In a stochastic environment differences in attitudes towards risk, expectations, and time preferences, cause investors to evaluate the potential outcome of investment decisions differently.

Thus, there does not exist an objective criteria to guide the management in its investment planning. In addition, the independence of real and financial decisions becomes questionable. The consensus in economic literature seems to be that the management

should be guided in its decisions by the interests of its shareholders. This statement has been interpreted to mean that the management should maximize either (i) the market value of the firm, or (ii) the expected utility of profit.

The expected utility of profit criterion is not unambiguous. Whose utility function is to be maximized? It certainly cannot be the utility function of all shareholders. If the utility function used represents the attitudes of the management, it cannot be said that the management acts on "behalf of the shareholders". The market-value criterion seems to be more reasonable. Higher market value is in the interest of all shareholders, even those who disagree with the policies of the management. They can still sell out for a higher price and re-invest elsewhere. However, this argument has been shown to be wrong [11]. One can construct an example in which all shareholders would choose a policy which does not maximize the value of the firm.

It is not clear whether the concensus on "the interests of the shareholders"is based on normative or descriptive grounds. One feels that the management is viewed as an instrument of the "invisible hand"; a concept so strongly imbedded in neoclassical thought. The management is appointed by shareholders and therefore it acts (or should act) so as to harmonize their conflicting interests. We can think of many reasons why the management does not (and should not) act in the interests of all shareholders.

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[1] One certainly does not expect the same unaminity in the workings of a properly elected democratic government.

In our model we will assume that the management is under the control of a few individuals who have a long-term commitment in the affairs of the firm,and who collectively own one share. In making its decisions, the management adopts a policy function which represents their interests. This policy function is analogous to the guidelines given to a democratic government by the leadership of its political party. In the case of the firm, it is the concensus of the controlling group in its commitment to the affairs of the firm.

It follows from our assumptions that the argument of the policy function is the dividends per share. Rationality is interpreted to mean that the management maximizes the expected utility of the (infinite) stream of dividends per share.

In our model there is also a Stock Market where titles of ownership of the capital stock of a firm are traded. Such an interpretation requires that (i) capital goods are not transferable among firms and that (ii) capital goods of different firms are distinct from each other. The work of Usawa in relation to Penrose effect is an excellent account of the unique characteristics incorporated in the capital assets of a particular firm [10]. The interpretation of Usawa necessitates the assumption that all firms are identical under certainty; under uncertainty, it implies that the probability distributionsof their returns are distinct.

From what it was said above it becomes clear that the Stock Market operates as a market under monopolistic competition. We do not show the existence of equilibrium; we simply assume it. We only borrow the concept of conjectural demand in order to study

the optimal policies of the management at the equilibrium prices of the Stock Market. These policies are studied under the assumption that the equilibrium in the Stock Market is the only guideline as to the future value of the capital assets of the firm.

In Section 3 we present the conditions which optimal policies satisfy for any concave, non-decreasing policy function.

These conditions imply that the optimal investment policies depend on the conjectural demand for the capital asset of the firm and are independent of policy functions of the management. Thus, there is separation of investment and financial decisions. The management chooses first the optimal investment policies, given its evaluation of the market. Then, it chooses the financial decisions to suite the interests of the controlling group. In other words, given the optimal investment, the management decides on the optimal finance in a manner equivalent to a dynamic portfolio problem.

Finally, in Section 4 we concentrate on constant elasticity policy functions and we prove the existence of optimal solutions.

1 MARKET ORGANIZATION

The resources of a community consist of capital and labour. The amount of resources available at the beginning of a period are used to produce a single commodity which is either consumed or transformed into capital. Production is carried out by firms with the following characteristics.

At the beginning of each period each firm possesses an amount of non-malleable capital goods. With the aid of these goods and labour hired at the beginning of the period, each firm produces a certain amount of the output of the commodity. Production is subject to constant returns to scale. The output produced during t is either sold to consumers or used as an investment good. Each firm uses part of the output in order to increase its own capital assets. Capital formation is costly and increases the productive capacity of the firm from period $(t+1)$ and on. Firms are perfect competitors in the commodity and labour markets. The current price of the commodity and of labour services are known with certainty. Future equilibrium prices are unknown.

The stock of capital which a firm possesses at the beginning of a period is owned by those who own the outstanding shares of the firm. A share is a title of ownership of a certain proportion of the capital stock of a firm, and entitles its owner to receive the same proportion of the net return from the operations of the firm during the period.

Firms can finance their capital formation in any combination of the following:

(i) by retaining part of the earnings which have accrued as a result of current production decisions, (ii) by issuing new shares which are sold in the Stock Market (to be introduced below), (iii) issuing one-period bonds. The current interest rate on bonds is non-random. Future rates are random.

The issue of new shares in period t is equivalent to the forward sale of part of the capital which the firm will have at the beginning of t+1. Because of this (i) those who buy new shares in period t (new shareholders, for simplicity) are entitled to dividends starting from period (t+1) and on, and (ii) the price of new shares reflects the expectations of the market about the future stream of returns which the capital assets of the firm will yield.

The Stock Market

An institution called the "Stock Market" convenes at the beginning of each period in order to facilitate transactions on bonds and shares. The supply of shares comes from the owners of outstanding shares and firms who wish to attract funds by issuing new equity. The demand for shares comes from individual investors. Firms do not hold shares of other firms. Both, firms and individuals may issue and hold bonds.

Firms may influence the equilibrium price of their shares. We assume that no individual investor, or group of investors, can affect the price of shares of any firm in the stock market.[2]

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[2] This does not deny the possibility that groups of individuals as shareholders may have an influence on the policies (and, thus, indirectly on the price of shares) of a firm.

The conditions under which firms may influence the price of their shares should be made clear. Shares have value because they are legal titles to the ownership of a certain amount of the firms' capital goods. These capital goods, in turn, incorporate the organizational, technological, and managerial characteristics of the firm which translate themselves in the ability of the firm to earn a net return per unit of capital invested. Suppose that all firms are identical with respect to technology and managerial abilities, and that there are no transaction and information costs. Under these conditions and the assumptions made about production and market organization, the price of the shares of all firms should be identical. However, when these characteristics differ among firms, their capital assets are not expected to yield the same returns; their capital assets become heterogeneous products which will command (indirectly through the price of shares) different prices in the stock market. In this case the firm, being the sole supplier of legal titles to the ownership of its (distinct from others) capital goods, can affect the price of its shares.

The Stock Market determines the optimum investment for each firm and the equilibrium prices and quantities of financial assets. The price of the output and of labour services are determined elsewhere (in the Commodity Market) and remain constant during the session of the Stock Market. Following Arrow and Hahn, we assume that the management of each firm perceives a demand curve (not necessarily the correct one) for its future capital assets,

given the prices and the quantities of the financial assets of other firms.[3] On the basis of this relation the management is assumed to decide on the optimum investment, the bonds to be issued, the price and the quantity of new shares to be supplied.[4] Furthermore, we assume that there exists equilibrium in the stock market.[5] Here we wish only to study the production and financial decisions which a firm has made at the equilibrium prices.[6]

2 THE MODEL

The following notations will be used in this model.

Y_t	output of period t.
N_t	the amount of labour hired at the beginning of t.
K_t	the capital stock at the beginning of t.
δ	the depreciation rate of capital stock.
$F(K_t,N_t)$	the one period production function
γ_t	the growth factor of K_t in t; i.e. $K_t\gamma_t$ is the gross addition to K_t in period t.

3 See [2] pp.151-167

4 The price of an outstanding share will be higher than the price of a new share by the amount of the dividends which the firms plans to distribute.

5 The existence of equilibrium under conditions (such as ours) of monopolistic competition, has been challenged, See [8].

6 To have equilibrium in the Stock Market and the Commodity Market, we must assume that the Commodity Market re-convenes after the closing of the stock market and determines a new equilibrium price of the commodity and of labour services. Then, the Stock Market operates anew to determine new equilibrium price of the financial assets. This process is repeated until both markets are cleared.

$K_t g(\gamma_t)$	the cost of $K_t \gamma_t$
p_t	the price of output
w_t	the real wage rate
R_t	the number of new shares issued in t
S_t	the number of outstanding shares at the beginning of t; i.e. $S_{t+1} = S_t + R_t$
X_t	total dividends distributed in t.
$x_t \equiv \frac{1}{S_t} X_t$	dividends per share
B_t	the number of one-dollar bonds purchased in t.
$L_t \equiv \frac{B_t}{S_{t+1}}$	
i_t	(one plus) the interest rate on bonds issued in t.
$\hat{q}_t$	the price of an outstanding share
q_t	the price of a new share issued in t.
$\bar{P}_t$	the implicit forward price in t of the capital goods which the firm will have in t+1.
$\hat{v}_t = \hat{v}_t(K_t)$	the value of the capital stock of period t.
u	the one-period policy function of the management.
$\alpha_t \equiv \frac{L_t}{\hat{q}_t - x_t}$	

Suppose that in period t the firm knows the equilibrium price of its new shares (q_t). The financial constraint which has to be met in any current period is that the sources of funds must equal the uses of funds, i.e. any production, investment, and financial decision in t must satisfy the equality.

$$(2.1) \qquad y_t + q_t R_t + B_{t-1} i_{t-1} = w_t N_t + K_t g(\gamma_t) + B_t + X_t$$

where q_t, B_t, w_t, B_{t-1}, X_t are values relative to the commodity price of period t. Furthermore, under the competitive conditions of our model:

$$(2.2) \qquad q_t S_{t+1} - B_t = \bar{p}_t K_{t+1},$$

i.e. the value of debt and equity at the closing of the stock market in t, must be equal to the forward price of the firms' capital stock of t+1.

Solve (2.1) for X_t and divide both sides of it by S_t. Then, add to both sides an expression equal to q_t (found from (2.2)). The resulting expression

$$\frac{1}{S_t} X_t + q_t \equiv x_t + q_t = \frac{1}{S_t} [y_t - w_t N_t - K_t g(\gamma_t) + \bar{p}_t K_{t+1} + B_{t-1} i_{t-1}]$$

is the value of an outstanding share in real terms (i.e. in terms of units of the commodity) in t, ($\hat{q}_t$). Rearrange terms to obtain:

$$S_t[\hat{q}_t + B_{t-1} i_{t-1}] = S_t[x_t + q_t - B_{t-1} i_{t-1}] = y_t - w_t N_t - K_t g(\gamma_t) + \bar{p}_t K_{t+1}$$

which says that the value of debt and equity outstanding at the beginning of t, must be equal to the net return from the operations

of the firm during t, plus the forward value of the firm's future capital goods. In other words, the RHS of the above expression is the value in real terms of the existing capital stock ($\hat{v}_t$); it is the price one would have to pay to buy the capital assets of the firm.

We summarize the above discussion with the following relation:

$$(2.3)\quad \hat{q}_t = x_t + q_t$$

$$= \frac{1}{S_t} [y_t - w_t N_t - K_t g(\gamma_t) + \bar{p}_t K_{t+1} + B_{t-1} i_{t-1}]$$

$$= \frac{1}{S_t} [\hat{v}_t(K_t) + B_{t-1} i_{t-1}]$$

where q_t is given by (2.2)

Similarly, the price of a share in terms of units of the commodity in t+1 is:

$$(2.4)\quad \hat{q}_{t+1} = \frac{1}{S_{t+1}} [\hat{v}_{t+1} (K_{t+1}) + B_t \frac{p_t i_t}{p_{t+1}}],$$

where $\frac{p_t i_t}{p_{t+1}}$ is the real rate of interest on bonds.

Solve (2.3) for $\frac{1}{S_{t+1}}$ with the aid of (2.2), and substitute in (2.4),

$$(2.5)\quad \hat{q}_{t+1} = [\hat{q}_t - x_t] \frac{\hat{v}_{t+1}}{\bar{p}_t K_{t+1}} + [\frac{p_t i_t}{p_{t+1}} - \frac{\hat{v}_{t+1}}{\bar{p}_t K_{t+1}}] L_t$$

where $L_t \equiv \frac{B_t}{S_{t+1}}$, $K_{t+1} = K_t (1+\gamma_t-\delta)$, and

$$(2.6)\quad \hat{q}_t = \frac{1}{S_t} [y_t - w_t N_t - K_t g(\gamma_t) + \bar{p}_t K_{t+1}] + L_{t-1} i_{t-1}$$

In (2.5) $\frac{\hat{v}_{t+1}}{\bar{p}_t K_{t+1}}$ is (one plus) the anticipated rate of return in t+1 of capital assets whose ownership was acquired in period t.

We will assume that the management of the firm behaves rationally if it follows a feasible production, investment, and finance policy which maximizes $E[\sum_{t=0}^{\infty} a^t u(x_t)/H_0]$ subject (2.5) and (2.6) where $E[./H_0]$ denotes the conditional expectation given the past and present information.

The maximization of the present value of the stream of dividends per share (rather than total dividends) intends to take into account the discrepancy between ownership and control of the firm In real life, the control of the firm-corporation is usually exercised by either a group of shareholders who more or less appoint the management, or by a strong managerial group. In many cases, these two groups are indistinguishable[7]. The average shareholder exercises little or no influence on the policies of the management. Agreements among "ordinary" shareholders to effectively voice their opinion with the management, are difficult for the following reasons: (a) the body of present shareholders is anonymous and changes every day, (b) a considerable amount of

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[7] This usually reflects the historical evolution of the corporation out of a family business who "went public" but where the "family" maintained the essential control.

technical expertise is required to form an educated opinion, (c) it requires a long-term committment in the affairs of the corporation (which is not in the nature of the average investor) to formulate sensible policy suggestions which, if implemented, will not endanger its profitability. (d) large (usually institutional) investors have wide diversified portfolios and serve their interests better by shifting funds among shares of different corporations, rather than devoting their energies into influencing the decision-making process of any one of them. After all, if a group of shareholders is strong enough, it will take over the control of the corporation instead of merely trying to influence the policies of the management.[8] For these reasons the average investor leaves the decision-making up to the management as long as he is reasonably satisfied with its policies.

In any case, we will assume that there is a group of individuals who controll the management of our firm. We define as one the number of shares that these individuals own collectively at $t=0$. We will also assume that the management may sell new shares to (or buy and destroy outstanding shares from) the general public. Share-splits are not permitted. Thus, the number of shares that the controlling group owns remains always equal to one. In addition we will assume that the maximum number of shares to be issued is bounded above (by $\bar{\bar{s}}$) so that the group will not loose the control of the management. Therefore, (a) $1 \leq S_t \leq \bar{\bar{s}}$ for all t, and (b) during any period the group owns $\frac{1}{S_t}$ of the capital assets of the firm and receives $\frac{X_t}{S_t} \equiv x_t$ of the total dividends. Hence, x_t (rather than X_t) is the appropriate argument of u. We make the following assumptions.

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8 Clearly such take-overs are not the goal of most institutional investors.

Assumption 1 $u(x_t)$ is a strictly concave, non-decreasing function of x_t for $x_t \geqq 0$, and $u'(o) = \infty$.

Assumption 2. P_t, i_t, w_t are random variables such that (i) w_t is identically and independently distributed over time and $\bar{w} \leqq w_t < \infty$ for some $\bar{w} > 0$. (ii) $G(p_{t+1}, i_{t+1}/h_0, h_1, \ldots h_t) = G(p_{t+1}, i_t/h_t)$ where $G(./h_0, h_1, \ldots, h_t)$ denotes conditional distribution given h_0, h_1, h_t. (iii) $Pr\{p_t \notin [\bar{\pi}, \bar{\bar{\pi}}]\} = 0$ for some $(\bar{\pi}, \bar{\bar{\pi}})$, $0 < \bar{\pi} < \bar{\bar{\pi}} < \infty$. (iv) $1 < \bar{i}_t \leqq i_t < \infty$

Assumption 3. $y_t = F(K_t, N_t)$ is homogeneous of degree one and such that $F_{KK} < 0$, $F_{NN} < 0$, $F_{KN} > 0$.

Assumption 4. $g(\gamma_t)$ is a strictly convex, non-decreasing function of γ_t, $\gamma_t \geqq 0$ and $g(0) = 0$, $g'(0) = 1$.

Definition 1: P_t is the set of (p_t, i_t, w_t) such that $Pr\{(p_t, i_t, w_t) \in P_t\} = 1$.

Let θ be a shift parameter.

Assumption 5. $\bar{p}_t = \bar{p}(K_{t+1}, p_t, i_t, w_t\ \theta)$ is a continuous function of $(p_t, i_t, w_t) \in P_t$, and such that $\bar{p}_t K_{t+1}$ is a strictly concave, non-decreasing function of K_{t+1}, $K_{t+1} \geqq 0$.

Assumption 6. For any t, $\hat{v}_{t+1} = r\bar{p}_t K_{t+1}$ where r is a random variable identically and independently distributed overtime. $Pr\{r \notin [\bar{r}, \bar{\bar{r}}]\} = 0$ for some $(\bar{r}, \bar{\bar{r}})$ such that $1 < \bar{r} < \bar{\bar{r}} < \infty$ and $\bar{r} < \bar{\bar{s}}$

Assumption 7 $Pr\{\dfrac{p_t i_t}{p_{t+1}} - r < 0 | H_0\} > 0$ and $Pr\{\dfrac{p_t i_t}{p_{t+1}} - r > 0 | H_0\} > 0$.

Assumption 8. Equity rules require that $\dfrac{\bar{p}_t K_{t+1}}{S_{t+1}} + L_t \geqq 0$.

Assumption 1 implies risk aversion and requires that the management considers it rational to distribute dividends, however small, in every period. Assumption 2 defines a Markov process in the formation of expectations with respect to future prices and interest rates and stationary process with respect to future real wage rates. Assumption 3 is the usual specification of a production function under constant returns to scale. Assumption 4 is the Penrose effect as specified by Uzawa [10]. Assumption 5 defines the (inverse of the) conjectural demand for the capital assets of the firm on the set of possible equilibrium prices of a period. The management, according to Assumption 5, believes that the equilibrium price p_t, i_t, w_t (which are beyond the control of any agent in the market) dictate the optimal policies of other firms. By making certain assumptions about their policies, the management derives the conjectural demand for the future capital assets of the firm. The shift parameter θ serves to adjust the perceived demand during the tatonnement process of the Stock Market. Assumption 7 requires that the rate of return on capital is not expected with certainty to be greater (or less) than the real rate of return on bonds. Assumption 8 sets an upper limit to the debt of the firm. Assumption 6 introduces a simplification which can be easily defended. In our model each firm differs from others with respect to organization, entrepreneurial skills, technological characteristics etc. All these aspects are incorporated into the entity called the "capital assets of a firm".

The optimal production investment, and finance decisions of a firm, are determined by the equilibrium in the commodity and stock markets. There, the different characteristics of each firm are taken into account (together with expectations, subjective evaluation of risk, time preferences etc. of the agents) in the determination of the equilibrium prices. These prices serve not only as a means of allocating the resources of the community where there are mostly needed, but also, as a means of punishment and reward for entrepreneurial decisions taken in the past. In particular, the (implicit) forward price of the firms' future capital stock is a "prize" awarded by the collective wisdom of the market for its unique characteristics.[9] And the price of these assets (as implied by the equilibrium in the stock market of this period) is assumed to be the only reliable method of gazing into a future which is full of uncertainties of any kind. By Assumption 6 (2.5) becomes:

$$\hat{q}_{t+1} = [\hat{q}_t - x_t]r + [\frac{p_t i_t}{p_{t+1}} - r]\ L_t \tag{2.7}$$

The model we have developed can be summarized as follows. At the beginning of t the vector $(K_t,\ S_t, L_{t-1})$ describes the "state" of the firm. Given a price vector $(p_t,\ i_t,\ w_t)$, the management (i) determines the optimum amount of labour services to be hired (N_t) and, thus, the optimal output (y_t), (ii) formulates its expectations with respect to future prices, (iii) perceives the demand for its future capital assets $\bar{p}(K_{t+1}, \theta)$ from certain assumptions it makes about the behaviour of other firms. For any value of the shift parameter (θ), the maximization of the objective

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9 Recall that we have assumed that no groups of individual investors, can affect the price of shares in the stock market and firms do not hold shares of other firms.

function determines the optimum amount of investment (γ_t) and finance (x_t, S_{t+1}, L_t) to be undertaken if (p_t, i_t, w_t) turns out to be the equilibrium price vector.Given $(\gamma_t, x_t, S_{t+1}, L_t)$the price of th shares isdetermined form (2.2).In its deliberations during t the management believes that in equilibrium, the forward price of its future capital assets is the only predictor of the value which these assets will actually have in t+1 . The above assumptions, (2.6) and (2.7) define the following relations for t=0,1,2,...

$$(2.8)\quad \hat{q}_t = \frac{1}{S_t}[y_t - w_t N_t - K_t g(\gamma_t) + \bar{p}_t K_{t+1}] + L_{t-1} i_{t-1}$$

where $K_{t+1} = K_t(1+\gamma_t-\delta)$.

$$(2.9)\quad \Pr\{(\hat{q}_t - x_t) r_t + (\frac{p_t i_t}{p_{t+1}}) - r) L_t \geqq 0 \mid H_0\} = 1$$

Constraint (2.9) requires that management considers only policies which do not lead to bankrupcy.

Furthermore, $\frac{1}{\bar{\bar{s}}} \leqq \frac{1}{S_{t+1}} \leqq 1$ imposes the additional constraint:

$$(2.10)\quad \frac{1}{\bar{\bar{s}}}\, \bar{p}_t K_{t+1} \leqq \hat{q}_t - x_t - L_t \leqq \bar{p}_t K_{t+1}, \text{ from } (2.3).$$

All together (2.8) - (2.10) define the set of feasible solutions $(y_t, N_t, \gamma_t, y_t, L_t)$, t=0,1,2,...

3. OPTIMALITY CONDITIONS.

In Section 2 we have defined the following maximization problem.

$$(3.1)\quad \text{maximize } E[\sum_{t=0}^{\infty} a^t u(x_t) \mid H_0]$$

subject to (2.8), (2.9), (2.10) and

$y_t \geqq 0$, $N_t \geqq 0$, $\gamma_t \geqq 0$, $x_t \geqq 0$, K_0= some positive constant, $L_0 = 0$.

It is convenient to approach the above maximization problem in two steps. First, the optimum (y_t, N_t) can be determined immediately: given K_t and w_t choose N_t so as to maximize $y_t - w_t N_t$. At the optimum, $y_t - w_t N_t$, $t=1,2,,,$ is equal to $K_t \rho_t$, where ρ_t is a random variable whose time path is a stationary process, and $\rho_t \geqq \bar{\rho}$ for some $\bar{\rho} \geqq 0$. This is a consequence of perfect competition in the labour market, constant returns to scale (Assumption 3), and the expectations we have postulated (Assumption 2(i)). Second, we can eliminate (y_t, N_t) from (2.8) - (2.10) and reformulate the problem as follows:

(3.3) maximize $E[\sum_{t=0}^{\infty} a^t u(x_t) | H_0]$

(3.4) Subject to $\Pr\{(\hat{q}_t - x_t) r_t + (\frac{p_t i_t}{p_t+1} - r) L_t \geqq 0 | H_0\} = 1,$

(3.5) $$\hat{q}_t = \frac{K_t}{S_t} [\rho_t - g(\gamma_t) + \bar{p}_t (1+\gamma_t-\delta)] + L_{t-1} i_{t-1},$$

(3.6) $$\frac{1}{\bar{\bar{s}}} \bar{p}_t K_{t+1} \leqq \hat{q}_t - x_t - L_t \leqq \bar{p}_t K_{t+1}, \text{ and}$$

$$x_t \geqq 0,\ \gamma_t \geqq 0,\ K_0 = \text{some positive constant}, L_0 = 0.$$

The set of feasible policies defined by (3.4) (3.7) is non-empty: Consider the policy: $g(\gamma_t) = \bar{\rho} - \varepsilon$, $x_t = (1 - \frac{1}{\bar{r}}) \cdot \hat{q}_t$, $L_t = 0$ for all t. Clearly (3.4) is satisfied. Furthermore, (3.6) and (3.7) become:

$$\frac{\bar{r}}{\bar{\bar{s}}} \leqq 1 + \frac{\varepsilon S_{t+1}}{(1+\gamma_t-\delta) S_t} \leqq \bar{r}$$

which is again satisfied for a sufficiently small ε, by Assumption 6.

The optimization problem presented above is similar in nature to the problem studied by Anastasopoulos and Konnias in [1]. The main difference between the two problems is the specification of the set of feasible solutions: in [1] there are no constraints equivalent to (3.5) and (3.6).[10] Because of this similarity, here we will summarise the common results and we will extend the proofs of [1] to cover constraints (3.5) and (3.6).

Let γ_t, $t=0,1,2,\ldots$, be a feasible investment policy and $\hat{q}_t$ the corresponding to γ_t value of the state variable given by (3.5) Then, as it is shown in [1], the optimal divident policy is in general, such that $0 < x_t < \hat{q}_t$.[11] However, when u is unbounded above and bounded below, a version of the St. Petersburg paradox may occur under certain conditions. The management may postpone paying dividends indefinitely in the hope that sometime in the distant future it will bring its shareholders bliss by distributing arbitrarily large dividends.

In any case $\hat{q}_t - x_t > 0$ and we can rewrite (2.7) and (2.10) as follows:

$$(3.8) \qquad \hat{q}_{t+1} = (\hat{q}_t - x_t) \; [r + (\frac{p_t i_t}{p_{t+1}} - r)\alpha_t]$$

$$\frac{1}{\bar{s}} \bar{p}_t \, K_{t+1} \leqq (\hat{q}_t - x_t) \; (1-\alpha_t) \leqq \bar{p}_t \, K_{t+1},$$

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10 The other formal difference is that in this paper r is a random variable while the corresponding variable in [1] is non-random. This difference is not significant; most proofs as in [1] when one uses the limits of r, $(\bar{r}, \bar{r})$ given in Assumption 6.

11 Lemma 1 of [1].

where $\alpha_t \equiv \dfrac{L_t}{q_t - \tilde{x}_t}$ is the (negative of the) debt-equity ratio of firm. We can reformulate the maximization problem (3.3) - (3.6) as follows:

$$(3.9)\qquad \text{maximize} \quad E[\sum_{t=0}^{\infty} a^t u(x_t)\,|H_0]$$

$$(3.10)\qquad \text{subject to } \Pr\{(\hat{q}_t - x_t)\ [r + (\frac{p_t i_t}{p_{t+1}} - r)\alpha_t] \geqq 0\,|H_0\} = 0$$

$$(3.11)\qquad \hat{q}_t = \frac{K_t}{S_t}\,[\rho_t - g(\gamma_t) + \bar{p}_t(1+\gamma_t - \delta)] + L_{t-1}\, i_{t-1}$$

$$(3.12)\qquad \bar{\alpha}_t \equiv 1 - \frac{\bar{p}_t\ K_{t+1}}{\hat{q}_t - x_t} \leqq \alpha_t \leqq 1 - \frac{\bar{p}_t K_{t+1}}{\bar{\bar{s}}(\hat{q}_t - x_t)} \equiv \bar{\bar{\alpha}}_t.$$

$\gamma_t \geqq 0$, $x_t \geqq 0$, K_0 = some positive constant, $L_0 = 0$.

It can be easily seen that the set of α_t which satisfies (3.10) is the closed and bounded interval $[\bar{e}_t, \bar{\bar{e}}_t]$ where

$$(3.13)\qquad \bar{e}_t = -\frac{1}{(p_t i_t/\bar{r}\bar{p}_{t+1})-1} < 0\ ,\quad \bar{\bar{e}}_t = \frac{1}{1-(p_t i_t/\bar{\bar{r}}\bar{\bar{p}}_{t+1})} > 1.$$

Assumption 7 guarantees that $\dfrac{p_t i_t}{\bar{r}\ \bar{p}_{t+1}} > 1$ and $\dfrac{p_t i_t}{\bar{\bar{r}}\ \bar{\bar{p}}_{t+1}} < 1$.

Theorem 1. Let $u(x_t)$ be bounded either above or below and let

$\gamma_t = \gamma(K_t, p_t, i_t, w_t)$, $x_t = x(\hat{q}_t, p_t, i_t)$, $L_t = L(\hat{q}_t, p_t, i_t)$

be a feasible policy which satisfies the sufficient conditions.

(i) $-g'(\gamma_t) + \bar{p}_t + \bar{p}_t' K_{t+1} \leqq 0$, where $\bar{p}_t^i \equiv \frac{\partial}{\partial K_{t+1}} \bar{p}_t$

(ii) $u'(x_t) - a\,E[u'(x_{t+1})\,\{r + (\frac{p_t i_t}{p_{t+1}} - r)\,\alpha_t\}\,|h_t] = 0$

(iii) $E[(\frac{p_t r_t}{p_{t+1}} - r)\;u'(x_{t+1})\,|h_t] \begin{cases} = 0 \text{ for } \alpha_t = \bar{\alpha}_t \\ = 0 \text{ for } \alpha_t\ \varepsilon\,(\underline{\underline{\alpha}}_t, \bar{\bar{\alpha}}_t \\ > 0 \text{ for } \alpha_t = \underline{\underline{\alpha}}_t \end{cases}$

(iv) $\lim_{t\to\infty} a^{t+1}\, E[u'(x_{t+1})\hat{q}_{t+1}\,|H_0]=0.$

Then, γ_t, x_t, L_t is optimal for the particular management of the firm.

Proof: To prove condition (i) of Theorem 1 consider any debt equity ratio in $(\max\{\bar{\alpha}_t, \bar{e}_t\}, \bar{\bar{\alpha}}_t)$, where $\bar{\bar{\alpha}}_t$ and $\bar{\bar{\alpha}}_t$ are defined in (3.12)[12]. Then, from (3.8) notice that $\hat{q}_{t+1}$ is an increasing function of $\hat{q}_t$ for all relizations of the random variables. Therefore, the optimum γ_t can be determined from (3.11) as a static maximization problem. The strict concavity of $\bar{p}_t\ K_{t+1}$ (Assumption 5) and the strict convexity of $g(\gamma_t)$ (Assumption 4) guarantee that the second order conditions for a maximum hold .

When $\alpha_t = \bar{\alpha}_t \equiv 1 - \dfrac{\bar{p}_t K_{t+1}}{\hat{q}_t - x_t}$, (3.8) becomes:

$$(3.14)\quad \hat{q}_{t+1} = (\hat{q}_t - x_t)\{\frac{p_t i_t}{p_{t+1}} + (r - \frac{p_t i_t}{p_{t+1}})b_t\}$$

12 It is easy to verify that $\bar{\bar{\alpha}}_t < \bar{\bar{\gamma}}_t$.

where $b_t \equiv \dfrac{\bar{p}_t K_{t+1}}{\hat{q}_t - x_t}$, and where our assumptions quarantee that

$$\Pr \{ (\hat{q}_t - x_t) \{ \frac{p_t i_t}{p_{t+1}} + (r - \frac{p_t i_t}{p_{t+1}}) b_t \} | H_0 \} = 1 .$$

Again $\hat{q}_{t+1}$ is an increasing function of $\hat{q}_t$ for all realizations of the random variable, and the optimum γ_t can be determined from (3.11) as before. A similar proof holds when $\alpha_t = \bar{\bar{\alpha}}_t$. The proof of conditions (ii)-(iv) is along the same lines as the proof of Theorem 1 of [1].

It should be stressed that the investment decisions of the management are interwoven with the operation of the stock market.[13] It appears that there is unanimity between management and shareholders on investment policies since the optimum γ_t does not depend on u. However $\bar{p}_t$ is a perceived demand and as such it incorporates the subjective evaluation of the management about the behaviour of other firms. Shareholders may have different perceptions. In a different way, the aggregate investment of a community (as determined by the equilibrium in the stock market) is profoundly influenced by the subjective views of those individuals who are managing its production units. This fact has several implications for empirical studies on investment and on stock market prices.

From Theorem 1 it follows that the management decides about its optimal policies in a two-step process. First, it chooses the optimum investment policy (given it conjectural demand) and then decides on the optimum financing of the investment (given the policy function u which represents the interests of the controlling

[13] For similar result in the case of a static model with uncertainty about the price of output see [6].

group). Thus we have arrived at a separation of "real" and "financial" decisions[14]; responsible for this result is our Assumptions 5, Assumptions 6, and the absence of risk of default implied by (3.10).

The conditions of Theorem 1 do not require that one considers policies which make the objective function finite. When only such policies are considered, it can be shown that they are necessary conditions for optimality. Furthermore, it can be shown that when u is homohetic, the optimal policies (x_t, L_t) are linear in $\hat{q}_t$.

4. CONSTANT ELASTICITY UTILITY FUNCTIONS.

Optimal financial policies do not necessarily exist for an arbitrary u. In this section we will concentrate on the class of constant elasticity utility functions.

(4.1) Let $u(x_t) = \frac{1}{1-\lambda} x^{1-\lambda} \qquad \lambda > 0, \lambda \neq 1,$

and consider the linear financial policy

(4.2) $x_t = c(p_t, i_t) \cdot \hat{q}_t \equiv \hat{q}_t c_t \,, \qquad L_t = \ell(p_t, i_t)\hat{q}_t \equiv \ell_t \cdot \hat{q}_t$

Using (4.2) the conditions (ii) - (iv) can be written as follows:

(4.3) $c_t^{-\lambda} - a(1-\lambda)(1 - c_t)^{-\lambda} E[c_{t+1}^{-\lambda} u(r + [\frac{p_t i_t}{p_{t+1}} - r]\alpha_t) | h_t] = 0$

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14 For similar results in the case of perfect capital markets see [4] ch.7 and [7] ch.7

$$(4.4)\quad E[(\frac{p_t i_t}{p_{t+1}} - r)c_{t+1}^{-\lambda} \quad u'(r + [\frac{p_t i_t}{p_{t+1}} - r]\alpha_t)|h_t] \begin{cases} < \text{ for } \alpha_t = \bar{\alpha}_t \\ = 0 \text{ for } \alpha_t \varepsilon (\bar{\alpha}_t \bar{\bar{\alpha}}_t) \\ > 0 \text{ for } \bar{\alpha}_t = \bar{\bar{\alpha}}_t \end{cases}$$

$$(4.5)\quad \lim_{t\to\infty} \quad E[u'(x_t)\ \hat{q}_t|H_0] = 0$$

We will show that given the optimum investment policy (condition (i) of Theorem 1), there exists a financial policy of the form (4.2) which satisfies (4.3) - (4.5) and, therefore, is optimal.

First, notice that the function

$$k_t \equiv E[c_{t+1}^{-\lambda} \ u(r + [\frac{p_t i_t}{p_{t+1}} - r]\alpha_t)|h_t]$$

is strictly concave in α_t, $\alpha_t \varepsilon [\bar{e}_t, \bar{\bar{e}}_t]$. Thus, for any c_{t+1}, $o < c_{t+1} < 1$, there exists an α_t in $[\bar{e}_t, \bar{\bar{e}}_t]$ which maximizes k_t.

Then, define the following:

Definition 2: $$k_t^* = \max_{\alpha_t} \quad E[c_{t+1}^{-\lambda} \ u(r + [\frac{p_t i_t}{p_{t+1}} - r]\alpha_t)|h_t]$$
$$= E[\ c_{t+1}^{-\lambda} \ u(r + [\frac{p_t i_t}{p_{t+1}} - r]\alpha_t^*)|h_t].$$

Definition 3: $$\hat{k}_t = a(1-\lambda) \max_{\alpha_t} E[u(r + [\frac{p_t r_t}{p_{t+1}} - r]\alpha_t)|h_t].$$

$$\text{for } \alpha_t \varepsilon [\bar{e}_t, \bar{\bar{e}}_t]$$

In Definition 3, $\hat{k}_t$ is the management's evaluation of financial opportunities given its time preference and policy the function u. Alternative opportunities are evaluated on the basis of how well

they will serve the dividend policy in the future provided that the best combination of these opportunities is choosen in the present.[15] The assumption which is made below with respect to $\hat{k}_t$ eliminates the occurance of the St.Petersburg paradox discussed earlier.

<u>Theorem 2</u> Let $u(x_t) = \frac{1}{1-\lambda} x_t^{1-\lambda}$, $\lambda>0$, $\lambda\neq 1$. Assume either (i) $o<\lambda<1$. and there exists $\varepsilon_t>0$ with $\sum_{t=0}^{\infty} \varepsilon_t=\infty$, such that $\hat{k}_t < 1-\varepsilon_t$ for all $(p_t, r_t) \varepsilon P_t$, $t=1,2,\ldots$, or (ii) $\lambda> 1$. Then, for any given $(p_t, i_t) \varepsilon A_t$ there exists a feasible policy $\tilde{x}_t = \tilde{c}_t \cdot \hat{q}_t$, $\tilde{L}_t = \tilde{\ell}_t \cdot \hat{q}_t$ which is optimal and such that $\underline{o}<\tilde{c}_t< 1$, and $\tilde{\ell}_t = \tilde{\alpha}_t(1-\tilde{c}_t)$, $\tilde{\alpha}_t \varepsilon [\max \{\bar{e}_t, \bar{\alpha}_t\}, \bar{\bar{\alpha}}_t]$, $\bar{\alpha}_t \equiv 1 - \frac{\bar{p}_t K_{t+1}}{\hat{q}_t(1-\tilde{c}_t)}$ $\bar{\bar{\alpha}}_t \equiv 1 - \frac{\bar{p}_t K_{t+1}}{\hat{q}_t(1-\tilde{c}_t)\bar{\bar{s}}}$.

It should be noticed that not all debt equity- ratios in $[\bar{e}_t, \bar{\bar{e}}_t]$ are available to the firm; the constraint (3.12) restricts α_t in $[\bar{\alpha}_t, \bar{\bar{\alpha}}_t$ which is a subset of $[\bar{e}_t, \bar{\bar{e}}_t]$. More important, the boundary points $\bar{\alpha}_t$ and $\bar{\bar{\alpha}}_t$ contain the decision variable $x_t = c_t\tilde{q}_t$. In (1) the existence of solutions is proven for α_t in $[\bar{e}_t, \bar{\bar{e}}_t]$, and it is shown that

$$(4.6)\quad c_t^* = \{1 + \{a(1-\lambda)\alpha_t^*\}^{1/\lambda}\}^{-1}$$

The novelty of our Theorem 2 is that it claims that existence of solutions in $[\max\{\bar{\alpha}_t, \bar{\gamma}_t\}, \bar{\bar{\alpha}}_t]$

<u>Proof</u>: The proof is given by an iteration procedure which uses as a subroutine the iteration procedure explained in the proof of Theorem 2 of [1]. Let $I_t^i, i = 1,2,\ldots$ be a family of subsets of

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15 For an extended discussion see [1], 267-268.

of $[\bar{e}_t, \bar{\bar{e}}_t]$. Define the relations:

$$(4.7) \qquad \alpha_t^{*i}(p_t,i_t),\ k_t^{*i}(p_t,\ i_t),\ c_t^{*i}(p_t,\ i_t),\ \alpha_t \epsilon I_t^i, (p_t,i_t) \epsilon P_t,$$
$$i=1,2,\ldots$$

to be the optimal solution which results from the iteration procedure of Theorem 2 of [1]. Given a particular $(p_t, i_t) \epsilon P_t$ find c_t^{*i} from (4.7) and define:

$$(4.8) \qquad \bar{\alpha}_t^i = 1 - \frac{\bar{p}_t K_{t+1}}{\hat{q}_t[1-c_t^{*i}]} \quad , \qquad \bar{\bar{\alpha}}_t^i = 1 - \frac{\bar{p}_t K_{t+1}}{\hat{q}_t[1-c_t^{*i}]\bar{\bar{s}}}$$

and $I_t^{i+1} = [\max\{\bar{e}_t, \bar{\alpha}_t^i\}, \bar{\bar{\alpha}}_t^i]$.

Let $I^1 = [\bar{e}_t, \bar{\bar{e}}_t]$. Derive $\alpha_t^{*i}(p_t,i_t)$, $k_t^{*i}(p_t,i_t)$ $c_t^{*i}(p_t,i_t)$ for $\alpha_t \epsilon I^1$. For a given (p_t,i_t) derive α_t^{*1}, and I_t^2 following the procedure explained above. If $\alpha_t^{*1} \epsilon [\max\{\bar{e}_t, \bar{\alpha}_t^1\}, \bar{\bar{\alpha}}_t] \equiv I_t^2$ the theorem has been proven. If α_t^{*1} does not lie I_t^2, it must be that $\max\{\bar{e}_t, \bar{\alpha}_t^1\} < \tilde{\alpha}_t < \bar{\bar{\alpha}}_t^1$. Then repeat the iteration procedure of Theorem 2 of [1] and derive the optimal solution of the type (4.7) (with a superscript 2) for all $(p_t, r_t) \epsilon P_t$ and $\alpha_t \epsilon I_t^2$. For the same (p_t, i_t) as in the first step, derive α_t^{*2} and I_t^3. If $x_t^{*2} \in I_t^3$ the theorem is proven. If not, it must be that $\max\{\bar{e}_t, \bar{\alpha}_t^2\} < \tilde{\alpha}_t < \bar{\bar{\alpha}}_t^2$, and the same procuedure should be repeated. Notice that

$$\max\{\bar{e}_t, \bar{\alpha}_t^1\} \leqq \max\{\bar{e}_t, \bar{\alpha}_t^2\} < \ldots\ldots \leqq \bar{\bar{\alpha}}_t^2 \leqq \bar{\bar{\alpha}}_t^1 < 1$$

From the uniqueness of solutions and the continuity of the functions in (4.7) we conclude that for some i there exists an interval I_t^i such that $\alpha_t^{*i-1} \epsilon\, I_t^i$. The last step of the above interation procedure determines also $\tilde{c}_t$ and ℓ_t for the particular (p_t, i_t) vector of period t. This complete the proof.

Corallary Let γ_t be the optimum investment policy which corresponds to a price vector (p_t, i_t, w_t). Then, the optimum financing of investment through bonds is independent of γ_t and of the time preference of the management (a).

Proof: It follows from the proof of Theorem 2 (in particular (4.8) that the optimum debt-equity ratio $(\tilde{\alpha}_t)$ depends on $\frac{p_t K_{t+1}}{\hat{q}_t}$ which in turn is determined by the optimum investment policy γ_t. Furthermore.

$$\frac{1}{S_{t+1}} = \tilde{\sigma}_t q_t = [\hat{q}_t - \tilde{x}_t - L_t] \frac{1}{\bar{p}_t K_{t+1}} \quad \text{from (2.2) and (2.3)}$$

$$= \hat{q}_t (1-\ell_t)\ (1-\tilde{c}_t) \frac{1}{\bar{p}_t K_{t+1}}$$

Take the ratio: $\frac{L_t}{1/S_{t+1}} = B_t = \frac{\ell_t}{\tilde{\sigma}_t}$ This completes the proof.

One could proceed and study the implications of a change in price expectations and of present prices on the optimal policies of the management. The results though depend crucially on the particular form of the conjectural demand $\bar{p}_t$. The specification of the functional form of $\bar{p}_t$ involves numerous theoretical problems and it will not be attempted here.

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